Properties of Rational Functions

A ***rational function*** is a function of the form $R(x)=\dfrac{p(x)}{q(x)}$, where p and q are polynomial functions and q is not the zero polynomial.

Finding the Domain of a Rational Function
The domain of a rational function is the set of all real numbers except those for which the denominator q is 0.

Examples: Find the domain of each rational function.

a) $R(x)=\dfrac{2x^2-3}{x+3}$ b) $R(x)=\dfrac{x+3}{x^2-9}$ c) $R(x)=\dfrac{x^2}{x^2+7x+12}$

★ **Note:** It is important to understand that $R(x)=\dfrac{x+3}{x^2-9}$ and $f(x)=\dfrac{1}{x-3}$ are not the same. They have different domains. Their graphs are nearly identical, but the graph of the first function has a hole in it at $x=-3,$ while the graph of the second function does not.

A rational function $R(x)=\dfrac{p(x)}{q(x)}$ is in ***lowest terms*** if $p(x)$ and $q(x)$ have no common factors.

Finding the Intercepts of a Rational Function
★ **Warning:** Always write the rational function in lowest terms before finding the *x*-intercepts of the graph. Otherwise, you may end up listing values that are actually holes in the graph. When finding the *y*-intercepts, remember that if zero is not in the domain, there is no *y*-intercept

To find the *x*-intercepts (real zeros) of the graph of a rational function, we set $R(x)=0$ and solve for x.

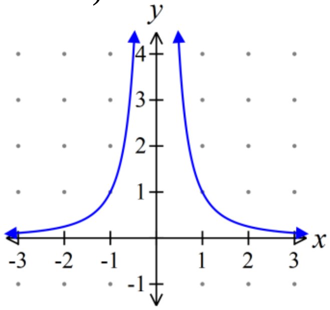
Notice that if $R(x)=\dfrac{p(x)}{q(x)}=0,$ $p(x)$ must equal zero. It is the numerator that tells us about the *x*-intercepts.

- To find the ***x*-intercepts** (real zeros), simplify the rational function and set the numerator equal to zero.
- To find the ***y*-intercept,** plug $x=0$ into either the simplified or the unsimplified function. If zero is not in the domain of the function, there is no *x*-intercept.

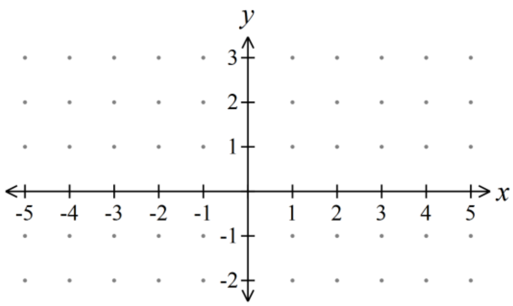
Examples: Find the *x*- and *y*- intercepts of each rational function.

a) $R(x)=\dfrac{x+2}{x^2-16}$ b) $R(x)=\dfrac{x^2-2x-35}{x^2+11x+30}$ c) $R(x)=\dfrac{x}{x^2+8x}$

Example: Analyze the graph of $H(x)=\dfrac{1}{x^2}$. (Find the domain, range, intercepts, and any other important features).



Example: Using transformations, graph $R(x)=\dfrac{1}{(x+2)^2}-1$.



Asymptotes

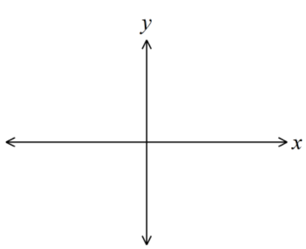
In the previous example, notice that as the values of x become more negative, that is, as x becomes ***unbounded in the negative direction*** $\left(x \to -\infty, \text{ read "as } x \text{ approaches negative infinity"}\right)$ the values of $R(x)$ approach -1. We conclude the following:

1. As $x \to -\infty$, the values of $R(x) \to -1$.
2. As $x \to -2$, the values of $R(x) \to \infty$.
3. As $x \to \infty$, the values of $R(x) \to -1$.

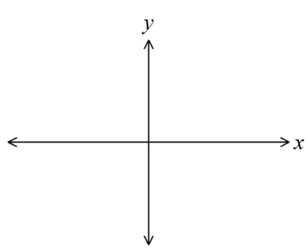
The graph goes near, but does not reach the lines $x=-2$ and $y=-1$. These lines are called ***asymptotes*.**

Let R denote a function.

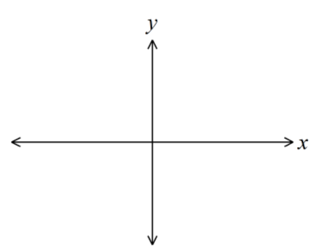
- If, as $x \to -\infty$ or as $x \to \infty$, the values of $R(x)$ approach some fixed number L, then the line $y=L$ is a ***horizontal asymptote*** of the graph of R.
- If, as x approaches some number c, the values of $R(x)$ approach ∞ or $-\infty$, then the line $x=c$ is a ***vertical asymptote*** of the graph of R.



$\lim\limits_{x\to\infty} R(x)=L$

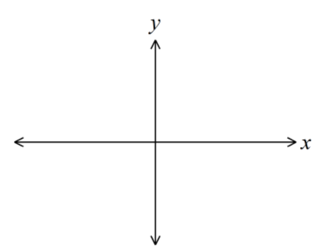


$\lim\limits_{x\to-\infty} R(x)=L$



$\lim\limits_{x\to c^-} R(x)=\infty$

$\lim\limits_{x\to c^+} R(x)=\infty$



$\lim\limits_{x\to c^-} R(x)=-\infty$

$\lim\limits_{x\to c^+} R(x)=\infty$

There is a third type of asymptote called an ***oblique asymptote.*** An oblique asymptote occurs when the graph's end behavior follows an oblique line (a linear function of the form $y = ax + b,\ a \neq 0$). A graph may also approach some function such as a parabola.

★ **Note:** The graph of a function may intersect a horizontal or oblique asymptote at some other point, but the graph will never intersect a vertical asymptote.

Find the Vertical Asymptotes of a Rational Function

A rational function $R(x) = \dfrac{p(x)}{q(x)}$, in *lowest terms* will have a vertical asymptote at $x = r$ if r is a real zero of the denominator q. That is, if $x - r$ is a factor of the denominator q of the rational function in lowest terms, it will have the vertical asymptote $x = r$.

- **Warning:** If the rational function is not in lowest terms, this theorem will result in an incorrect listing of vertical asymptotes.

Example: Find the vertical asymptotes, if any, of the graph of each rational function.

a) $R(x) = \dfrac{x+1}{x^2 - 9}$

b) $R(x) = \dfrac{x+2}{x^2 - 3x - 10}$

c) $R(x) = \dfrac{x}{x^3 + 10x^2 + 9x}$

d) $R(x) = \dfrac{x^2}{x^2 + 4}$

Find the Horizontal or Oblique Asymptotes of a Rational Function

- If a rational function $R(x)$ is ***proper,*** that is *if the degree of the numerator is less than the degree of the denominator*, then the line $y = 0$ is a horizontal asymptote.
- If a rational function $R(x) = \dfrac{p(x)}{q(x)}$ is ***improper,*** that is, if the *degree of the numerator is greater than or equal to the degree of the denominator*, we can use long division to write the rational function as the sum of a polynomial $f(x)$ plus the proper rational function $\dfrac{r(x)}{q(x)}$. As $x \to -\infty$ or as $x \to \infty$, $\dfrac{r(x)}{q(x)} \to 0$, and the graph of $R(x)$ approaches the graph of $f(x)$.
  - The possibilities include:
    1. If $f(x) = b$, a constant, the line $y = b$ is a horizontal asymptote of the graph of R.
    2. If $f(x) = ax + b,\ a \neq 0$, the line $y = ax + b$ is an oblique (slanted) asymptote of the graph of R.
    3. In all other cases, the graph of R approaches the graph of f, and there are no horizontal or oblique asymptotes.

Examples: Find the horizontal or oblique asymptotes, if any, of the graph of the function.

a) $R(x)=\dfrac{x+5}{x^2-3x-10}$

b) $H(x)=\dfrac{6x^2-2x+5}{3x^2-2}$

c) $H(x)=\dfrac{8x^3+2x^2-6}{2x^2-3}$

d) $H(x)=\dfrac{3x^4+4x^2-7}{x^2-2x+5}$

Summary:

Given a rational function $R(x) = \frac{p(x)}{q(x)}$,

1. The **domain** is found by setting the denominator not equal to zero and solving. (Do not simplify first).
2. The ***x*-intercepts** are found by setting the numerator to zero and solving. (Simplify first).
3. The ***y*-intercept** is found by setting $x = 0$ and simplifying. (You can do this before or after simplifying the function). Remember, if zero is not in the domain, the function has no *y*-intercept.
4. **Vertical asymptotes** are found by setting the denominator to zero and solving (Simplify first).
5. **Horizontal or Oblique Asymptotes:**
    a) If **proper,** where $\textit{degree of numerator} < \textit{degree of denominator}$, the graph has a **horizontal asymptote** at $y = 0$.
    b) If **improper**, where $\textit{degree of numerator} = \textit{degree of denominator}$, the graph has a **horizontal asymptote** at the line $y = \frac{\text{leading coefficient of the numerator}}{\text{leading coefficient of the denominator}}$.
    c) If **improper**, where $\textit{degree of numerator} > \textit{degree of denominator (by one)}$, the graph has an **oblique asymptote** of $y = f(x)$ found by performing long division.
    d) If **improper**, where $\textit{degree of numerator} > \textit{degree of denominator (by more than one)}$, then R has neither a horizontal nor an oblique asymptote, but the end behavior can be determined using long division.